NUMERICAL SOLUTION OF A HEAT-TRANSFER PROBLEM

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Using numerical calculations, a study is made of the relation between the mean temperature of a surface and the heat flux, in cooling a solid with a stream of liquid.

The present paper describes results of a numerical investigation on the Minsk-2 computer of the problem considered in [1]. A number of variants have been solved with different values of β, K_1, K_2 and $\bar{\alpha}|_{\bar{y}=R/d} = 0$ for the case R = d, $h_{\bar{s}} = \bar{d}/10$ [2]. Calculation of the asymptote at the point (0, 0) was accomplished by replacing a five-point scheme by variable coefficients $\bar{A}_{i,j}$ and $\bar{B}_{i,j}$ at the points shown in Table 2 [2], apart from the point (0, 0), where the scheme assumes a relation between the points (0, 0) and (0, 1).

In accordance with the technique of [3], the upper boundary with respect to $\bar{\xi}$ was chosen to be 2.6, and the boundary condition was taken as

$$\left(\frac{\partial u}{\partial \bar{\xi}} - \alpha_f u\right)\Big|_{\bar{\xi}=2,6} = 0 \qquad (\alpha_f = 2.831) \tag{1}$$

and $h_{\bar{\xi}} = 0.2$. In choosing the boundary in the liquid, which is the boundary condition for large $\bar{\xi}$, we must take care that the mesh scheme does not become unstable because of the symmetric approximation to the term $(1/3)\bar{\xi}^2(\partial T^{(f)}/\partial \bar{\xi})$ in the heat-transfer equation. The large value of dimensionless pitch $h_{\bar{\xi}}$ in comparison with $h_{\bar{s}}$ is due to the smoother behavior of $u = T^{(f)}$ with respect to $\bar{\xi}$ in the region corresponding to the liquid.

The method of calculation involves introduction of successive corrections, which are found by the fractional pitch method [4, 5].

The original system of equations can be written as

$$Lu = f \tag{2}$$

the symmetric part of the operator L being positive definite. The structure of the operator L, acting in vector space, and with its components numbered by the two subscripts, is determined in accordance with [2]. For application of the fractional pitch method the operator L is represented in the form

$$L = L_1 + L_2,$$

where L_1 is the operator given by the formulas:

$$\begin{gathered}\{L_1 u^{(s)}\}_{i,j} = \bar{A}_{i,j}\,(u_j^{(s)})_{\bar{x}_i^{2}}, \quad j = 0,\ i = 1, 2, \ldots, 10 \\ \text{and } j = 1, 2, \ldots, 10, \quad i = 0, 1, \ldots, 10; \\ \{L_1 u^{(f)}\}_{i,j} = 0, \quad j = 0,\ i = 0 \\ \text{and } j = -1, -2, -3, \ldots, -13, \quad i = 0, 1, \ldots, 10;\end{gathered} \tag{3}$$

with i = 0 and i = 10

$$u^{(s)}_{i-1,j} = u^{(s)}_{i+1,j}$$

and L_2

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$$\{L_2u^{(s)}\}_{i,j}=\bar{B}_{i,j}\left(u_i^{(s)}\right)_{\bar{y}_j^2}+\bar{\beta}\,u_{i,j}^{(s)},\quad i=0,\ 1,\ \dots,\ 10,\quad j=10,\ 9,\ \dots,\ 1;$$

$$\{L_2u^{(s)}\}_{i,j}=\frac{\bar{B}_{i,j}}{h_{\bar{s}}^2}\left[-\frac{1}{K_2h_{\bar{\xi}}\,\bar{\eta}_i^{1/3}\bar{D}_i}\,u_{i,j-1}^{(f)}-\left(2-\frac{1}{K_2h_{\bar{\xi}}\,\bar{\eta}_i^{1/3}\bar{D}_i}+\frac{K_1h_{\bar{\xi}}\,\bar{\eta}_i^{-1/3}}{2K_2\bar{D}_i}\right)u_{i,j}^{(s)}+\left(1-\frac{\bar{C}_i}{\bar{D}_i}\right)u_{i,j+1}^{(s)}\right]+\bar{\beta}\,u_{i,j}^{(s)},\quad i=1,\ 2,\ \dots,\ 10,\quad j=0;$$

$$\{L_2u^{(s)}\}_{0,0}=\frac{1}{1+\dfrac{0{,}932478h_{\bar{s}}^{2/3}}{K_2}+\dfrac{0{,}297575h_{\bar{s}}^{4/3}}{K_2^2}}\;\frac{1}{h_{\bar{s}}^{2/3}}\,u_{0,1}^{(s)}-\frac{1}{h_{\bar{s}}^{2/3}}\,u_{0,0}^{(s)};\tag{4}$$

$$\{L_2u^{(f)}\}_{i,j}=\left(u_i^{(f)}\right)_{\bar{\xi}_j^2}-\frac{j^2h_{\bar{\xi}}^2}{3}\left(u_i^{(f)}\right)_{2\bar{\xi}_j},\qquad i=0,\ j=-1,\ -2,\ \dots,\ -13;$$

$$\{L_2u^{(f)}\}_{i,j}=\frac{1}{2}\left[\left(u_i^{(f)}\right)_{\bar{\xi}_j^2}-\frac{j^2h_{\bar{\xi}}^2}{3}\left(u_i^{(f)}\right)_{2\bar{\xi}_j}+K_1\bar{\eta}_i^{-2/3}u_{i,j}^{(f)}\right]+\frac{jh_{\bar{\xi}}}{h_{\bar{\eta}}}\,\bar{\eta}_{i-1}^{\rm cp}\,u_{i,j}^{(f)}+\frac{1}{2}\left[\left(u_{i-1}^{(f)}\right)_{\bar{\xi}_j^2}-\frac{j^2h_{\bar{\xi}}^2}{3}\left(u_{i-1}^{(f)}\right)_{2\bar{\xi}_j}+K_1\bar{\eta}_{i-1}^{-2/3}u_{i-1,j}^{(f)}\right]-\frac{jh_{\bar{\xi}}}{h_{\bar{\eta}}}\,\bar{\eta}_{i-1}^{\rm cp}\,u_{i-1,j}^{(f)},$$

$$i=1,\ 2,\ \dots,\ 10,\quad j=-1,\ -2,\ \dots,\ -13,$$

where, for j = −13, we must include in the formulas $u_{i,j-1}^{(f)}$, using Eq. (1) in the form $(u_i^{(f)})_{2\bar{\xi}_j}|_{j=-13}=\alpha_f\cdot u_{i,j}^{(f)}|_{j=-13}$, and for j = 10 $u_{i,j+1}^{(s)}$, using the boundary condition at the lower boundary of the body [1], $(u_i^{(s)})_{2\bar{y}_j}|_{j=10}=q_id/\lambda_s$, by relating the term $-2q_id/\lambda_sh_{\bar{s}}$ in $\{L_2u^{(s)}\}_{i,10}$ to f in Eq. (2), which gives $f_{i,10}=-2q_id/\lambda_sh_{\bar{s}}$. The quantity $\bar{\eta}_i^{\rm av}$ was determined in [2].

It is clear that the problem of calculating $(E-\tau L_i)^{-1}f$ with a diagonal in the chosen representation of the vectors by the positive operator τ can be solved effectively by direct methods (for L_2 it is decomposed into a number of problems with tridiagonal matrices, and for L_1 it reduces to the solution, using an implicit scheme, of a degenerate parabolic equation, where $\bar{x}$ plays the role of the time coordinate).

Instead of the problem of (2) we shall solve the equivalent problem of

$$\Pi\,u\equiv\tau\,Lu=\tau\,f=g.\tag{2a}$$

The choice of the operator τ must further satisfy the requirements of rapid convergence of the method for specific values of the parameters, and simple convergence for as wide a range as possible of values of the parameters. An investigation and a justification of analogous transformations of the original equations was made in [6] by Lebedev in the case of kinetic equations, and also in [7]. At least we should require that there should be no obstacles to application of internal pitch steps of the scheme. At the same time the calculation pursued the objective of investigating the application of the scheme in the noncommutative case of the scheme [8].

The scheme consists of seeking corrections to some approximation u_l

$$\begin{aligned}u_{l+1}&=u_l+\delta\,u_l,\\ \tau\,f_l&=\Pi\,u_{l-1}-g\quad(\Pi_i=\tau\,L_i,\ i=1,\ 2),\\ v_{l,k}&=(E-\Pi_2)^{-1}(u_{l,k-1}+\tau\,f_l),\\ u_{l,k}&=(E-\Pi_1)^{-1}v_{l,k},\\ u_{l,0}&=0,\ \ \delta\,u_l=u_{l,k_l}.\end{aligned}\tag{5}$$

For a specific numerical process here we must assign sets $\{k_l\}$ and $\tau_{l,k}$ $(1\le k\le k_l)$.

In choosing τ we should take account of the following considerations, based partially on investigation of cases close to commutative. Small values of $\tau_{l,k}$, of the order of the reciprocal of the main eigenvalue of the operator Π, take a good account of error components corresponding to large eigenvalues, even in the case when they are used once. Larger values of $\tau_{l,k}$ take better account of error components corresponding to smaller eigenvalues of the operator Π, but, besides the poor calculation of components with large eigenvalues, an additional, substantial error appears in these components in the noncommutative case. To avoid this $\tau_{l,k}$ was chosen to diminish with k, roughly in geometric progression, and the smallest of these, for fixed L, were of the order of the inverse of the spectral limits of the operator Π. The diagonal operator $\tau = \{\tau_{i,j}\}$ was chosen from the following considerations. First, in performing an operation with respect to j, transition to an unstable condition must not occur, i.e., such that in the operator $W = (E - \tau L_2)$, the matrix elements $W_{i,j;i,j} \geq |W_{i,j;i,j+1}| + |W_{i,j;i,j-1}|$, which, of course, imposes a restriction only when $\bar{\beta} > 0$. For j > 0 (inside the body) with $\tau_{i,j}$ we obtain a quantity of order $1/\bar{\beta}$, and for j = 0 the estimate takes the form

$$\tau_{l,k}(i,\ j) = \frac{1}{\bar{\beta}\left(1 - \frac{\bar{B}_{i,j}}{h_{\bar{s}}^2} \cdot \frac{K_1' h_{\bar{\xi}} \bar{\eta}_i^{1/3}}{2K_2 \bar{D}_i}\right)} \quad (K_1 = K_1' \bar{\beta}). \tag{6}$$

In the liquid (j < 0) the original problem is parabolic, so that here there are no upward restrictions on $\tau_{i,j}$ and we can even take $\tau_{i,j} = +\infty$ ($\tau \approx 10^4$ was taken in the calculations) since, in cases when the corresponding problem of transforming $(E - \tau L_2)$ encounters a singularity, the results of the calculation are inapplicable because the approximation mesh becomes unsuitable. On the other hand, we must not choose $\tau_{i,j}$ appreciably larger than unity when j > 0, since then the conditions imposed in transforming $(E - \tau L_1)$ deteriorate severely. Since we should first work on the convergence of the method, we must choose a smaller value of the corresponding estimates. In addition, near (i = 0, j = 0), because of the rapid change in the coefficients of the mesh equation, smaller values of $\tilde{\tau}_{i,j}$ were chosen, and, in fact, a factor of the type $2^{-3+(i+j)}$ was added to $\tau_{i,j}$ for i ≥ 0, j ≥ 0 to bring it up to 1 (at the point i = 0, j = 0 no correction was applied). Since the calculation used only the computational process described above, suitable for nonnegative symmetric parts of the operators L_1 and L_2, the only restriction is on $\bar{\beta}$, for which the restriction was applied.

For a preliminary estimate of this characteristic value of $\bar{\beta}$ the following model problem inside the body was used:

$$\frac{\partial^2 T^{(s)}}{\partial \bar{y}^2} + \bar{\beta}\, T^{(s)} = 0, \quad 0 < \bar{y} < \frac{R}{d},$$

$$\left.\frac{\partial T^{(s)}}{\partial \bar{y}}\right|_{\bar{y}=0} = \frac{3}{3{,}715 K_2} \left. T^{(s)} \right|_{\bar{y}=0},$$

$$\left.\frac{\partial T^{(s)}}{\partial \bar{y}}\right|_{\bar{y}=\frac{R}{d}} = -\frac{\alpha d}{\lambda_s} \left. T^{(s)} \right|_{\bar{y}=\frac{R}{d}},$$

The coefficient in the boundary condition $\bar{y} = 0$ was determined using the solution of the corresponding model in the liquid [1], with a constant temperature boundary. The value of the characteristic $\bar{\beta}$, denoted below by $\bar{\bar{\beta}}$, was estimated by the moment method, in the case when the liquid heat capacity was neglected, to be:

$$\bar{\bar{\beta}} = \frac{3}{1 + 3.715 K_2} \quad \left(\alpha|_{\bar{y}=\frac{R}{d}} = 0\right). \tag{7}$$

All the estimates in Eq. (6) were taken with $\bar{\beta}$ replaced by $\bar{\bar{\beta}}$. The set $k_l \tau_{l,k}$ was chosen from the considerations described earlier and according to the following rule as the result of the fractional calculations. All the $k_l = 5$.

$$\tau_{l,k} = \zeta \tau_l \tau_k, \tag{8}$$

where the sequence τ_l is periodic with period (1; 0.6; 0.3), τ_k is the set (1; 0.6; 0.3; 0.1; 0.03). The value of ζ was chosen empirically. It was found most favorable to take $\zeta = 2$. In cases close to critical it was necessary to reduce ζ. As an initial approximation everywhere the zero vector was used, although we should have used solutions for adjacent values of the parameters. In Eq. (2) $\{f\}_{i,j} = 0$, apart from $\{f\}_{i,10} = -2q_i d / \lambda_s h_{\bar{s}}$, at the lower boundary of the body R/d.

In order to use the results we need to know the average temperature of the surface and the flux through it, i.e.,

$$\frac{1}{d}\int_0^d \lambda_f \frac{\partial T^{(f)}}{\partial y} dx = \frac{1}{d}\int_0^d \lambda_s \frac{\partial T^{(s)}}{\partial y} dx = q.$$

TABLE 1. Values of Mean Boundary Temperature, Dimensionless Heat Flux, and Heat-Transfer Coefficient

K_2	K_1'	$\{\bar{\beta}\}$	$T_{av}^{(s)}$	p_{av}	p_c	α
5	5	−0,153	2,5390	0,5706	0,5583	0,2248
		−0,0765	3,6134	0,7037	0,6934	0,1948
		0	6,1969	1,0047	1	0,1621
		+0,03825	9,5921	1,3876	1,3908	0,1447
		+0,0765	21,0870	2,6635	2,6951	0,1263
	1	−0,153	2,9159	0,5118		0,1755
		−0,0765	3,9847	0,6729		0,1689
		0	6,1969	1,0047	1	0,1621
		+0,03825	8,5135	1,3512		0,1587
		+0,0765	13,4889	2,0942		0,1553
	0,5	−0,765	0,8874	0,1727	0,1642	0,1947
		−0,153	2,9755	0,5025	0,4947	0,1689
		−0,0765	4,0387	0,6685	0,6616	0,1655
		0	6,1969	1,0047	1	0,1621
		+0,03825	8,3978	1,3472	1,3446	0,1604
		+0,0765	12,9314	2,0524	2,0548	0,1587
1	5	−0,636	0,3180	0,5963	0,5809	1,8754
		−0,318	0,5133	0,7263	0,7118	1,4150
		−0,159	0,7260	0,8278	0,8158	1,1403
		0	1,2226	1,0044	1	0,8215
		+0,159	3,8085	1,6814	1,7250	0,4415
	1	−0,636	0,4712	0,5089	0,5000	1,0802
		−0,318	0,6980	0,6666	0,6586	0,9550
		0	1,2226	1,0044	1	0,8215
		+0,159	1,8590	1,3965	1,3974	0,7512
		+0,318	3,6237	2,4582	2,4755	0,6784
		+0,4452	12,6327	7,8091	7,9130	0,6182
	0,5	−0,636	0,5099	0,4868	0,4795	0,9548
		−0,318	0,7359	0,6543	0,6477	0,8892
		0	1,2226	1,0044	1	0,8215
		+0,159	1,7559	1,3814	1,3801	0,7868
		+0,318	2,9808	2,2398	2,2462	0,7514
	0	−6,36	0,0492	0,0401		0,8159
		−3,18	0,1323	0,1082		0,8183
		−0,636	0,5590	0,4589		0,8208
		−0,318	0,7797	0,6402		0,8211
1	0	0	1,2226	1,0044	1	0,8215
		+0,159	1,6651	1,3682		0,8217
		+0,318	2,5449	2,0916		0,8219
0,5	5	−0,525	0,2047	0,7070	0,6929	3,4539
		−0,2625	0,3089	0,8193	0,8071	2,6524
		0	0,6042	1,0034	1	1,6607
		+0,2625	8,9808	3,4257	3,7353	0,3815
	1	−1,05	0,1975	0,4880		2,4712
		−0,525	0,3126	0,6520		2,0855
		0	0,6042	1,0034	1	1,6607
		+0,2625	1,0000	1,4317		1,4317
		+0,525	2,3734	2,8256		1,1905
	0,5	−1,05	0,2234	0,4657		2,0847
		−0,525	0,3398	0,6381		1,8779
		0	0,6042	1,0034	1	1,6607
		+0,2625	0,9105	1,4095		1,5480
		+0,525	1,6745	2,3988		1,4325
	0,05	−16,8	0,0054	0,0125		2,3093
		−10,5	0,0151	0,0314		2,0748
		−5,25	0,0494	0,0925		1,8706
		−1,05	0,2567	0,4372		1,7029
		0	0,6042	1,0034	1	1,6607
		+0,525	1,3378	2,1936		1,6396
	0	−10,5	0,0164	0,0270	0,0251	1,6433
		−5,25	0,0521	0,0861	0,0832	1,6505
		−1,05	0,2613	0,4332	0,4295	1,6583
		−0,525	0,3741	0,6208	0,6170	1,6595
		0	0,6042	1,0034	1	1,6607
		+0,2625	0,8375	1,3914	1,3887	1,6614
		+0,525	1,3093	2,1762	2,1753	1,6620

Since these quantities (the normal derivative $\partial T/\partial y$ and the function T itself) do not depend smoothly on x, to increase the accuracy we must use quadratic formulas, accurate asymptotically for these quantities. With the same degree of accuracy as the mesh equations approximate the differential equations, it is sufficient to use the quadratic formulas of trapezoid type. To find the average temperature we must formulate the integral sums

TABLE 2. Values of Coefficients and Roots of the Polynomials P_1 and P_2 and Residues in the Representation $1/\alpha(\bar{\beta}) = \sum_i (A_i/(\bar{\beta} - \bar{\beta}_i))$

n	$\{\bar{\beta}\}$	i	c_i	d_i	$\bar{\beta}_i^{(1)}$	$\bar{\beta}_i^{(2)}$	A_i	A_0
				$K'_1=1;\ K_2=1$				
3	—0,636	0	+1	+1,2173				
	—0,318	1	—0,9733	—0,5411	+1,567	+3,927	—1,601	
	0	2	+0,2259	+0,05886	+3,384	+5,267	+0,2037	
	+0,159	3	—0,0076		+24,88		—6,374	
	+0,318							
	+0,4452							
2	—0,318	0	+1	+1,2173				
	0	1	—0,1667	+0,4408	+1,563	—2,105	—1,570	+0,2159
	+0,159	2	—0,3027	—0,06534	—2,114	+8,851	—0,0057	
	+0,318							
	+0,4452							
	0	0	+1	+1,2173				
	+0,159	1	—0,65254	—0,15065	+1,5616	+8,0802	—1,5633	
	+0,318	2	+0,007797		+82,129		—17,759	
	+0,4452							
1	+0,159	0	+1	+1,217327				+0,21043
	+0,318	1	—0,639029	—0,134472	+1,5649	+9,0527	—1,5757	
	+0,4452							
1	+0,318	0	+1	+1,2064561				
	+0,4452	1	—0,5709971		+1,7513		—2,1129	
				$K'_1=5;\ K_2=0,5$				
2	—0,525	0	+1	+0,6021493				
	—0,2625	1	—3,1127689	—0,3349830	+0,32790	+1,7976	—0,16484	
	0	2	+0,1922853		+15,860		—1,5773	
	+0,2625							
1	—0,2625	+0	+1	+0,6021493				+0,09540
	0	1	—3,0453373	—0,2905297	+0,32837	+2,0726	—0,16640	
	+0,2625							
1	+0	0	+1	+0,6021493				
	+0,2625	1	—2,9345148		+0,34077		—0,20520	
				$K'_1=5;\ K_2=5$				
2	—0,153	0	+1	+6,1682				+1,0281
	—0,0765	1	—6,8663	—25,3879	+0,2787	+0,2797	—0,46103	
	0	2	+11,5991	+11,9250	+0,3333	+1,8493	—1,1192	
	+0,03825							
	+0,0765							
2	—0,0765	0	+1	+6,1682				
	0	1	—3,7210	—5,9898	+0,2918	+1,0298	—1,4107	
	+0,03825	2	+1,0067		+3,4043		—4,5390	
	+0,0765							
1	0	0	+1	+6,168179				+1,15157
	+0,03825	1	—3,378876	—3,891010	+0,295956	+1,585238	—1,484698	
	+0,765							
1	+0,03825	0	+1	+6,134674				
	+0,0765	1	—2,942717		+0,339822		—2,084698	
				$K'_1=0,5;\ K_2=5$				
2	—0,0765	0	+1	+6,168				
	0	1	—12,945	—78,163	+0,07892	+0,078914	+0,00004	
	+0,03825	2	+3,465		+3,657		—22,558	
	+0,0765							
1	0	0	+1	+6,1682				+1,8309
	+0,03825	1	—0,3873	—0,7091	+2,5820	+8,6987	—11,1990	
	+0,0765							
1	+0,03825	0	+1	+6,167606				
	+0,0765	1	—0,275940		+3,623982		—22,35129	

$$\sum_{i=0}^{\tilde{d}/h_{\bar{s}}} N_i u_{i,0}^{(s)k_l,l} \tag{9}$$

such that the functions u, continuous in each of the segments $ih_{\bar{s}}$, $(i + 1)h_{\bar{s}}$ are linear combinations of the main terms asymptotically, i.e., are combinations of the type $m_0 + m_1\bar{x}^{2/3}$. As a result we obtain:

$$N_0=\alpha_0,\ N_{\frac{\bar d}{h_{\bar s}}}=\beta_{\frac{\bar d}{h_{\bar s}}-1},\ N_i\Big|_{0<i<\frac{\bar d}{\bar s}}=\alpha_i+\beta_{i-1}\qquad(\bar d=1),$$

where

$$\alpha_i=h_{\bar s}\frac{(i+1)^{2/3}-\frac{3}{5}\left[(i+1)^{5/3}-i^{5/3}\right]}{(i+1)^{2/3}-i^{2/3}},$$

$$\beta_i=h_{\bar s}\left(1-\frac{\alpha_i}{h_{\bar s}}\right)\quad\text{for}\quad i=0,\ 1,\ \dots,\ 4,\tag{9a}$$

$$\alpha_i=\beta_i=\frac{h_{\bar s}}{2}\quad\text{for}\quad i=5,\ 6,\ \dots,\ \frac{\bar d}{h_{\bar s}}-1.$$

To calculate the mean flux we must use the quantity $\partial u^{(f)}/\partial\bar\xi|_{i,0}$, since it is finite and is obtained by numerical solution as

$$\frac{\partial u^{(f)}}{\partial\bar\xi}\Big|_{i,0}=\frac{1}{h_{\bar\xi}}\left[\left(1-\frac{K_1h_{\bar\xi}^2\bar\eta_i^{2/3}}{2}\right)u_{i,0}^{(f)}-u_{i,-1}^{(f)}\right],\ u_{0,0}^{(f)}=u_{0,0}^{(s)}.\tag{10a}$$

Then

$$\int\frac{\partial T^{(s)}}{\partial\bar y}d\bar x\Big|_{\bar y=0}=\frac{1}{K_2}\int\frac{\partial T^{(f)}}{\partial\bar\xi}\bar x^{-1/3}d\bar x\Big|_{\bar y=0}$$

and the integral sum is

$$\frac{1}{K_2}\sum_{i=0}^{\bar d/h_{\bar s}}M_i\frac{\partial u^{(s)k_l,l}}{\partial\bar\xi}\Big|_{i,0},\tag{10}$$

where the same asymptote with weight $\bar x^{-1/3}$ is valid for $\partial u^{(f)}/\partial\bar\xi$. This case is distinct from computation of the mean value of temperature only as regards the weight. The formulas are analogous for

$$M_0=\gamma_0,\ M_{\bar d/h_{\bar s}}=\delta_{\frac{\bar d}{h_{\bar s}}-1},\ M_i\Big|_{0<i<\frac{\bar d}{h_{\bar s}}}=\gamma_i+\delta_{i-1},\tag{10b}$$

where

$$\gamma_i=\delta_i=\frac{3}{4}h_{\bar s}^{2/3}\left[(i+1)^{2/3}-i^{2/3}\right]\quad\text{for}\quad i=0,\ 1,\ \dots,\ 4,$$

$$\gamma_i=\frac{h_{\bar s}^{2/3}}{2i^{1/3}},\quad\delta_i=\frac{h_{\bar s}^{2/3}}{2(i+1)^{1/3}}\quad\text{for}\quad i=5,\ 6,\ \dots,\ \frac{\bar d}{h_{\bar s}}-1.$$

The results of computer calculations for different values of K_1', K_2, $\bar\beta$ are shown in Table 1.

Calculation of the ratio between the averages was performed in some cases as an indirect check of the accuracy.

In averaging all the quantities with respect to $\bar x$, for the mean values of temperature inside the body we have the equation

$$\frac{\partial^2T^{(s)}}{\partial\bar y^2}+\bar\beta T^{(s)}=0,\quad 0<\bar y<\frac{R}{d}$$

with the condition

$$\frac{\partial T^{(s)}}{\partial\bar y}\Big|_{\bar y=\frac{R}{d}}=1$$

and the value calculated on the computer

$$T^{(s)}|_{\bar y=0}=T_{\text{av}}^{(s)}.$$

As the result of solution we have

$$\left.\frac{\partial T^{(s)}}{\partial \bar{y}}\right|_{\bar{y}=0} = -T_{av}^{(s)}\sqrt{-\bar{\beta}}\,\mathrm{th}\,(\sqrt{-\bar{\beta}}) + \frac{1}{\mathrm{ch}\,(\sqrt{-\bar{\beta}})},\quad \bar{\beta}<0$$

$$\left.\frac{\partial T^{(s)}}{\partial \bar{y}}\right|_{\bar{y}=0} = T_{av}^{(s)}\sqrt{\bar{\beta}}\,\mathrm{tg}\,(\sqrt{\bar{\beta}}) + \frac{1}{\cos\,(\sqrt{\bar{\beta}})},\quad \bar{\beta}>0, \tag{11}$$

which is compared with the value p_{av} calculated on the computer.

During the calculation values of local boundary temperatures were put out by the computer: they were mainly such that the asymptotic behavior for $\bar{x} \to 0$ was described quite accurately by the asymptote determined in [2], which also showed that the scheme is applicable for actual computations.

The results showed that convergence is realized for all values of the parameters for which the operator L_2 has a nonnegative symmetrical part. Here the rate of convergence is not appreciably less than the rate of approach to steady conditions of the corresponding nonsteady problem with equivalence of $\tau \sim \Sigma\tau_{l,k}$, since the elements $\tau_{i,j}$ were mainly of order unity. To obtain a solution of the problem for values of the parameters other than in this region, even when the solution exists and also for values close to critical, when convergence is severely decelerated, we can recommend a combination of the above scheme with moment methods, which make it possible to eliminate the main error components which increase or very slowly decrease when using the iteration scheme of Eq. (5).

One question arising in describing nonsteady phenomena in problems similar to that examined is the question of the possible use of average values. The equation inside the body in the problem considered assumed the average to be

$$T^{(s)}(\bar{x},\ \bar{y}) \to \frac{1}{\bar{d}}\int_0^{\bar{d}} T^{(s)}(\bar{x},\ \bar{y})\,d\bar{x} = \overline{T^{(s)}(\bar{y})},$$

$$\frac{1}{\bar{d}}\int_0^{\bar{d}} \frac{\partial T^{(s)}(\bar{x},\ \bar{y})}{\partial \bar{y}}\,d\bar{x} = \overline{\frac{\partial T^{(s)}(\bar{y})}{\partial \bar{y}}},$$

and the nonsteady equation has the form

$$\frac{\partial \overline{T^{(s)}}}{\partial \bar{t}} = \frac{\partial^2 \overline{T^{(s)}}}{\partial \bar{y}^2} + \bar{Q}(\bar{y},\ \bar{t}).$$

Regarding the boundary conditions at the solid–liquid boundary, here the average does not give an exact correspondence only between the average values. The average flux involves also other characteristics of the distribution of surface temperature, due to the presence of a term with a first derivative with respect to $\bar{x}$ in the heat-transfer equation in the liquid. For $\bar{y}$ = R/d we shall assume an exact derivation of the average values.

It is interesting to consider the question of approximating to the boundary condition for the solid with $\bar{y}$ = 0. By assigning a suitable initial temperature distribution at $\bar{t}$ = 0, we can obtain very arbitrary heat-transfer relations between the average quantities and, without isolating any class of processes, cannot deal with average quantities without resorting to solution of the complete equations.

As the class of required solutions we shall choose those which are sufficiently smooth in time, when we can consider that, as functions of time, all the quantities considered are approximated by finite linear combinations of exponential and polynomial functions. Here we try to investigate the application of the technique of [9], and shall study singularities of approximation to the boundary relation by means of differential relations of the type $P_1(\partial/\partial\bar{t})\tilde{T} - P_2(\partial/\partial\bar{t})p$ with equal degrees of polynomials P_1 and P_2, or when the degree of P_1 is one larger than necessary to have a starting point for further investigation.

We shall first consider certain peculiarities of the qualitative behavior of the exact boundary relation. The question arises of the uniqueness of describing the process with fixed $\bar{\beta}$, i.e., of the dependence of the effective heat-transfer coefficient on R/d, $\underline{\alpha}|_{R/d}$, $q(\bar{x})$ [2]. This relation exists, but, when R/d → ∞ and $\int q(\bar{x})d\bar{x} \neq 0$ it ceases to be single-valued for $\bar{\beta} < 0$. But if $\bar{\beta} > 0$, then the deviation of $q(\bar{x})$ from constancy can have an appreciable influence, since, for sufficiently large $\bar{\beta}$, solving the problem inside the body, components of the solution of type $\cos k\pi\bar{x}$, oscillatory with respect to $\bar{x}$, are not damped, and for $\int q(\bar{x})d\bar{x} = 0$ they are very appreciable. The question of the accuracy of estimating these must be examined specially.

We shall discuss the case of $K_1'\bar{\beta} \to -\infty$. It is not difficult to check that

$$\alpha(\bar{\beta}) \to \frac{\sqrt{-K_1'\bar{\beta}}}{K_2}. \tag{12}$$

The effect of K_2 on $\alpha(\bar{\beta})$ is quite evident and consists of a reduction of α with increase of K_2.

The case of large negative values of $K_1'\bar{\beta}$ was used as a basic approximation in [1], where attention was focused on the qualitative picture of the process and on convective transfer. This limiting case is typical in that the approximation by differential relations gives the same behavior as the approximating rational functions with respect to $\bar{\beta}$, as is obtained when the convective component is ignored. Even for mean values of K_1' an increase of α and of $-\bar{\beta}$ is observed, but, for very small K_1', α decreases with $-\beta$, due to the fact that the steady equations are in fact considered in the liquid, so that α increases in transition to boundary values of temperature which increase faster with respect to $\bar{x}$, and an increase in $-\beta$ leads to their becoming equal.

To make a complete investigation of the spectral properties of the boundary relation we must know the dependence of $\alpha(\bar{\beta})$ also for imaginary $\bar{\beta}$, but we confine ourselves only to rational approximations to $\alpha(\bar{\beta})$, constructed using calculated values for certain real $\bar{\beta}$.

For the approximation

$$\frac{1}{\alpha(\bar{\beta})} = \frac{P_2(\bar{\beta})}{P_1(\bar{\beta})} = \frac{\sum_i d_i\bar{\beta}^i}{\sum_i c_i\bar{\beta}^i} = A_0 + \sum_i \frac{A_i}{\bar{\beta}-\bar{\beta}_i} \tag{13}$$

Table 2 shows values of the coefficients of polynomials P_1 and P_2, their zeros and residues when $1/\alpha(\bar{\beta})$ is represented as the sum of simple fractions. Although these constructions require high accuracy in the original data, and the accuracy is insufficient for $h_{\bar{S}} \sim 0.1\bar{d}$, nevertheless, to explain the behavior of the approximations, we used the results of the calculations, since they correspond, in fact, to a similar lattice problem, i.e., to a model for which a mesh scheme is an accurate description. The problem enters as a degenerate case into the class of elliptical and parabolic mixed-composition problems considered, for which approximations of actual quantities are constructed using differential relations.

In the graph of $\{\bar{\beta}\}$ (Table 2) we indicate the set $\bar{\beta}$ for which values of α from Table 1 coincide with those calculated by the approximate formula. While the approximation of $1/\alpha(\bar{\beta})$ is constructed from $(2n + 1)$ points, the degrees of the polynomials P_1 and P_2 are the same, equal to n; in the case of 2n points the degree of P_1 is n and that of P_2 is $(n - 1)$.

It is important to note in Table 2 that, in contrast with the cases given in [9], the residues can have different signs, and that the zeros of the numerator and denominator need not alternate. In these cases $K_1' = 1$, $K_2 = 1$, $n = 2$ and $K_1' = 0.5$, $K_2 = 5$, $n = 2$ degeneracy is, in fact, observed, since the small residues $A_1 = -0.0057$ and 0.00004 can be replaced by zeros in the limits of accuracy of solution of the problem (even in the mesh). In the case $K_1' = 5$, $K_2 = 5$, $n = 2$ we observe the phenomenon of transition of two close roots of the denominator and of one root of the numerator adjacent to them in the expansion

$$\frac{1}{\alpha(\bar{\beta})} = \frac{\Sigma\, d_i\bar{\beta}^i}{\Sigma\, c_i\bar{\beta}^i}$$

into one root of the denominator upon interpolation with respect to 5 and 4 points, respectively, in terms of $\bar{\beta}$. In these variants, where $\alpha(\bar{\beta})$ varies only slightly, rational approximations do not give well-posed problems, and in addition, systems for coefficients of the polynomials $P_1(\bar{\beta})$ and $P_2(\bar{\beta})$ become so badly conditioned that they do not allow us to determine even a smooth root of $P_1(\bar{\beta})$.

As far as physical use of the zeros and poles of $1/\alpha(\bar{\beta})$ is concerned, only the first pole and residue describe the actual process with sufficient accuracy (such as the rate of cooling of a surface, when a liquid cannot give up the heat obtained in the previous time interval, and, in fact, no heat transfer is obtained). The remaining poles and residues here, besides the fact that they are not sufficiently accurate to describe the real process in the degree of approximation considered, refer to cases with variable-sign boundary temperatures and must depend strongly on details of the behavior of the equations inside the solid.

NOTATION

$u^{(s)}, u^{(f)}$ are the corrections to temperatures in the body and in the liquid, respectively;

$T^{(s)}_{\bar{t}=0} = T^{(f)}_{\bar{t}=0} = 0$; $(u_j)_{\bar{x}^2} = (u_{i-1,j} - 2u_{i,j} + u_{i+1,j})/h^2_{\bar{s}}$; $(u_i)_{\bar{y}^2} = (u_{i,j-1} - 2u_{i,j} + u_{i,j+1})/h^2_{\bar{s}}$; $(u_i)_{2\bar{\xi}j} = (u_{i,j+1} - u_{i,j-1})/2h_{\bar{\xi}}$; $(u_i)_{2\bar{y}_j} = (u_{i,j+1} - u_{i,j-1})/2h_{\bar{s}}$;

T_{av}, p_{av}, α are the values of the mean temperature, dimensionless flux, and heat-transfer coefficient at the solid–liquid boundary, as obtained from the computer;

$\underline{p}_c$ is the value of the control stream;

$\bar{t} = a_s t/d^2$ is the dimensionless time;

$c_i, d_i, \bar{\beta}_i^{(1)}, \bar{\beta}_i^{(2)}$ are the values of the coefficients and roots of the polynomials P_1 and P_2, respectively;

A_0, A_i are the integral part and the residues in representing $1/\alpha(\bar{\beta})$ in the form of simple fractions.

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